INEQUALITIES FOR DISCRETE HIGHER ORDER CONVEX FUNCTIONS: A SURVEY

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Abstract. If a function is n-times differentiable in an interval, then it is said to be convex of order n, if its derivatives of order n are positive. An equivalent formulation requires that all divided differences of order n of the function are positive. Higher order convex functions have been defined and extensively studied already in the first half of the 20th century and a monograph, published by Popoviciu in 1945, summarizes the most important results obtained until then. They attracted much interest later on as well and various inequalities have been proved, where Josip Pečarić is one of the main contributors. Higher order convex functions play important role in the Chebyshev-Markov moment problem, where a functional, acting on an unknown probability distribution, is to be minimized or maximized subject to moment constraints. The notion has a number of variants and generalizations. One of the most important among them is (higher order) convexity with respect to a Chebyshev system. In this context the original notion means convexity with respect to the system of a finite number of consecutive powers of the variable. One further important variant is obtained if we restrict the definition of the function to a discrete set and then define higher order convexity by divided differences. There are a number of applications where such functions come up in a natural way and no continuous counterparts exist. Discrete higher order convexity came to prominence by the discovery of the author, at the end of the 1980s that a class of sharp probability inequalities can be obtained as optimum values of linear programming problems, where the input data are binomial moments of unknown probability distributions. In this talk we summarize the inequality type results that have been obtained during the past twenty years in connection with discrete higher order convex functions. The utilized mathematical tools are primarily interpolation theory, linear programming and graph theory. We formulate LP's, where the matrices of the equality constraints are discrete Chebyshev systems and derive inequalities in closed forms, based on binomial or power moments, if the number of them is relatively small. If the number of moments is large, then we present inequalities in algorithmic forms, i.e., we design special finite algorithms that provide us with the numerical bounds in the inequalities as optimum values of optimization problems. Recent results in connection with factorization of Vandermonde matrices help us to obtain new results and give more elegant derivations to older ones. We will also study the relationship between the inequalities for smooth and discrete higher order convex functions. Those belonging to the first category can sometimes be derived by the use of the formulas given for the discrete case. Finally, we mention a number of applications in reliability theory.