INEQUALITIES INVOLVING MULTIDIMENSIONAL GRAM DETERMINANTS

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Abstract. Let $L_1, \dots, L_p$ be finite-dimensional (real or complex) vector spaces whose dimensionalities are $n_1, \dots, n_p$, and let $\Phi = \Phi(\vec{x}_1, \dots, \vec{x}_p)$ be a multilinear form defined on $L_1 \times \dots \times L_p$. For arbitrarily given families $X^{(\nu)} \subseteq L_\nu, \nu = 1, \dots, p$, where $X^{(\nu)} = \left\{ \vec{x}^{(\nu)}_{l_\nu} \right\}_{i_\nu = 1, \dots, n_\nu}$, we define a p-dimensional Gram matrix $\Gamma_\Phi = \|\gamma_{l_1 \dots l_p}\|$, where $\gamma_{l_1 \dots l_p} = \Phi(\vec{x}^{(1)}_{l_1}, \dots, \vec{x}^{(p)}_{l_p})$, and $\vec{x}^{(r)}_{l_r} \in X^{(r)}$.

For a given p-dimensional matrix $A = \|A_{i_1, \dots, i_p}\|, \quad i_r = 1, \dots, n_r$, and for arbitrarily given $(0,1)$-sequence $\sigma = (\sigma_1, \dots, \sigma_p)$ with even number of ones, we consider a p-dimensional determinant $|A|^{(\sigma)}$ (we call it σ-determinant, or determinant of signature σ):

$$|A|^{(\sigma)} = \sum_{\tau = (\tau_1, \dots, \tau_p)} (-1)^{\sum \sigma_k I_k} \prod_{i=1}^{m} A_{\tau_1(i), \dots, \tau_p(i)},$$

where $\tau_k \in S_{m, n_k},\ m = \min(n_r),\ S_{m,n}$ be the set of m-permutations over the set $\{1, \dots, n_k\}$, and I_k be number of inversions in τ_k.

We study properties of σ-determinants of the matrices Γ_Φ — the Gram σ-determinants associated with Φ — denoted by $G^{(\sigma)}_\Phi(X^{(1)}, \dots, X^{(p)})$.
Using generalized versions of Binet-Cauchy's theorem, we prove several determinantal identities for p-dimensional Gram determinants, $G^{(\sigma)}_\Phi(X^{(1)}, \dots, X^{(p)})$ both in general case of arbitrary Φ and in different particular cases. One of useful properties following from that identities is the positivity of p-dimensional gramians of even type that yields a family of inequalities generalizing many of classical and more recent results.