OPERATOR GEOMETRIC MEAN AND ITS USE

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Abstract. For positive invertible operators A and B on a Hilbert space, the operator geometric mean is defined by Ando as

$$A \sharp B = \max\{X \geq 0; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\}.$$

Note that if A is invertible, then it is expressed as

$$A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}},$$

and is a unique positive solution of the Riccati equation

$$XA^{-1}X = B.$$

Moreover, it corresponds to the operator monotonicity of $t^{\frac{1}{2}}$, i.e.,

$$A \geq B \geq 0 \implies A^{\frac{1}{2}} \geq B^{\frac{1}{2}}.$$

In succession with this, the Kubo-Ando theory on operator means induces the α-geometric mean

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}},$$

which corresponds to the Löwner-Heinz inequality. One of the most interesting inequalities on this is the Ando-Hiai inequality

$$A \sharp_\alpha B \leq 1 \implies A^r \sharp_\alpha B^r \leq 1 \quad (r \geq 1).$$

In this talk, we propose generalizations of it and give conceptual application of it.