

EULER HARMONIC IDENTITIES AND MOMENTS OF REAL BOREL MEASURES

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ABSTRACT. Let $M[a, b]$ be the Banach space of all real Borel measures on $[a, b]$ with the total variation norm. For $\mu \in M[a, b]$ define function $\check{\mu}_n : [a, b] \rightarrow \mathbf{R}$, $n \geq 1$, by $\check{\mu}_n(t) = \frac{1}{(n-1)!} \int_{[a,t]} (t-s)^{n-1} d\mu(s)$. We also write $e_n(x, \mu) = \int_{[a,b]} (s-x)^n d\mu(s)$, $n \geq 0$, $x \in [a, b]$ for the n -th x -centered moment of μ . A sequence of functions $P_n : [a, b] \rightarrow \mathbf{R}$, $n \geq 1$, is called **μ -harmonic sequence of functions** on $[a, b]$ if $P'_n(t) = P_{n-1}(t)$, $n \geq 2$, and $P_1(t) = c + \check{\mu}_1(t)$, $t \in [a, b]$, for some $c \in \mathbf{R}$. Define function $K_n : [a, b] \times [a, b] \rightarrow \mathbf{R}$, $n \geq 1$, by

$$K_n(x, t) = \begin{cases} P_n(t), & a \leq t \leq x, \\ P_n(t) + \frac{(-1)^n}{(n-1)!} e_{n-1}(t, \mu), & x < t \leq b \end{cases}$$

for $a \leq x < b$, while for $x = b$

$$K_n(b, t) = \begin{cases} P_n(t), & a \leq t < b, \\ P_n(b) - \check{\mu}_n(b), & t = b. \end{cases}$$

A sequence of polynomials $(Q_n, n \geq 1)$ is called **semiharmonic sequence of polynomials** if $Q'_n(t) = Q_{n-1}(t)$, $n \geq 2$, and $Q_1(t) = c$ for some $c \in \mathbf{R}$. The following theorem is the key result of our talk.

Theorem 1. *Let $f : [a, b] \rightarrow \mathbf{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n \geq 1$. Then for every semiharmonic sequence of polynomials $(Q_n, n \geq 1)$ we have $\int_{[a,b]} f(t) d\mu(t) + S_n(x) = (-1)^n \int_{[a,b]} K_n(x, t) df^{(n-1)}(t)$, for every $x \in [a, b]$, where*

$$S_n(x) = \sum_{k=1}^n (-1)^k \left[Q_k(b) f^{(k-1)}(b) - Q_k(a) f^{(k-1)}(a) \right] - \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k!} e_k(x, \mu).$$

Using Euler identity (1) we prove some of the Ostrowski type and Euler-Grüss type inequalities for functions of various classes. Among other results, we generalize some results from [M. Matić, J. Pečarić and N. Ujević, *Generalizations of weighted version of Ostrowski's inequality and some related results*, Journal of Inequalities and Applications, **5**, (2000), 639–666] as well as the recent results in [A. Čivljak, Lj. Dedić and M. Matić, *On Ostrowski and Euler-Grüss type inequalities involving measures*, Journal of Mathematical Inequalities, **1**, 1, (2007), 65–81].