THE LOCAL BEHAVIOUR OF VAN DER WAERDEN TYPE FUNCTIONS AND ESTIMATIONS OF APPROXIMATE CONVEXITY

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Abstract. Let $p \in]1,\infty[$ and $\Phi_p(x) = 2^p \sum_{n=0}^{\infty} \frac{(\mathrm{dist}(2^n x, \mathbb{Z}))^p}{2^n}$ $(x \in \mathbb{R})$. As it is stated in the following result of Házy and Páles (cf. [1, Theorem 6]), the function Φ_p plays a specific role in the theory of approximately convex functions.

Theorem 1. *Let $I \subset \mathbb{R}$ be an open interval and $\varepsilon \geq 0$. If $f : I \to \mathbb{R}$ fulfils the inequality*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \varepsilon |x-y|^p$$

for every $x, y \in I$ and f is locally bounded from above at a point, then f is locally bounded and the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \varepsilon \Phi_p(\lambda)|x-y|^p \tag{0.1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

In order to replace the coefficient $\Phi_p(\lambda)$ in (0.1) with a simpler expression, one may wish to determine the asymptotic magnitude of the function Φ_p around zero. This problem is left open by the authors [1, Remark 1]. Concerning this problem, we establish the following statement.

Theorem 2. *For every $1 < p \leq 2$, $q > 1$, $K > 0$, and $\delta > 0$, there exists a $\lambda \in \mathbb{R}$ such that $0 < \lambda < \delta$ and $\Phi_p(\lambda) > K\lambda^q$.*

This result is obtained from Theorem 1 and the following characterization of convex functions:

Theorem 3. *Let $I \subset \mathbb{R}$ be an open interval. If $f : I \to \mathbb{R}$ is locally bounded and there exist $p > 1$, $\varepsilon \geq 0$ such that f satisfies the inequality*

$$f\big(\lambda x + (1-\lambda)y\big) + f\big((1-\lambda)x + \lambda y\big) \leq f(x) + f(y) + \varepsilon\,(\lambda(1-\lambda)|x-y|)^p$$

for every $x, y \in I$ and $\lambda \in [0,1]$, then f is convex.

References

[1] A. Házy and Zs. Páles, *On approximately t-convex functions*, Publ. Math. Debrecen **66**/3–4 (2005), 489–501.